2 Four fundamental subspaces

1. Show that the range of every linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^m$, and show that every subspace of $\mathbb{R}^m$ is the range of some linear function.

2. Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ denote a given matrix. Show that following two claims hold. (i) $\mathrm{im}(A) =$ the space spanned by the columns of A (column space). (ii) $\mathrm{im}(A^\top) =$ the space spanned by the rows of A (row space).

Subspaces and Linear Functions For a linear function f mapping $\mathbb{R}^n$ into $\mathbb{R}^m$, let $\mathrm{im}(f)$ denote the *range* of f. That is, $\mathrm{im}(f) = \{f(\boldsymbol{x}) \mid \boldsymbol{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$ is the set of all "images" as $\boldsymbol{x}$ varies freely over $\mathbb{R}^n$.

The range of every linear function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a subspace of $\mathbb{R}^m$, and every subspace of $\mathbb{R}^m$ is the range of some linear function.

For this reason, subspaces of $\mathbb{R}^m$ are sometimes called *linear spaces*.

Range Spaces
The *range of a matrix* $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ is defined to be the subspace $\mathrm{im}(A)$ of $\mathbb{R}^m$ that is generated by the range of $f(\boldsymbol{x}) = A\boldsymbol{x}$. That is,

$$\mathrm{im}(A) = \{A\boldsymbol{x} \mid \boldsymbol{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

Similarly, the range of $A^\top$ is the subspace of $\mathbb{R}^n$ defined by

$$\mathrm{im}(A^\top) = \{A^\top\boldsymbol{y} \mid \boldsymbol{y} \in \mathbb{R}^m\} \subseteq \mathbb{R}^n.$$

Because $\mathrm{im}(A)$ is the set of all "images" of vectors $\boldsymbol{x} \in \mathbb{R}^m$ under transformation by A, some people call $\mathrm{im}(A)$ the *image space* of A.

Column and Row Spaces For $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, the following statements are true.
- $\mathrm{im}(A) =$ the space spanned by the columns of A (column space).
- $\mathrm{im}(A^\top) =$ the space spanned by the rows of A (row space).
- $\boldsymbol{b} \in \mathrm{im}(A) \Leftrightarrow \boldsymbol{b} = A\boldsymbol{x}$ for some $\boldsymbol{x}$.
- $\boldsymbol{a} \in \mathrm{im}(A^\top) \Leftrightarrow \boldsymbol{a}^\top = \boldsymbol{y}^\top A$ for some $\boldsymbol{y}^\top$.

3. Describe $\mathrm{im}(A)$ and $\mathrm{im}(A^\top)$ for $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$.

4. Show that for any two matrices A and B of the same shape the following (i), (ii) hold. (i) $\mathrm{im}(A^\top) = \mathrm{im}(B^\top)$ if and only if $A \overset{\text{row}}{\sim} B$. (ii) $\mathrm{im}(A) = \mathrm{im}(B)$ if and only if $A \overset{\text{col}}{\sim} B$.

Equal Ranges For two matrices A and B of the same shape:
- $\mathrm{im}(A^\top) = \mathrm{im}(B^\top)$ if and only if $A \overset{\text{row}}{\sim} B$.
- $\mathrm{im}(A) = \mathrm{im}(B)$ if and only if $A \overset{\text{col}}{\sim} B$.

5. Determine whether or not the following sets span the same subspace

$$\mathcal{A} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 1 \\ 4 \end{pmatrix} \right\}, \quad \mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right\}.$$

Spanning the Row Space and Range Let A be an $m \times n$, matrix, and let U be any row echelon form derived from A. Spanning sets for the row and column spaces are as follows:
- The nonzero rows of U span $\mathrm{im}(A^\top)$.
- The basic columns in A span $\mathrm{im}(A)$.

6. Determine spanning sets for $\mathrm{im}(A)$ and $\mathrm{im}(A^\top)$, where $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{bmatrix}$.

Nullspace
- For an $m \times n$ matrix A, the set $\ker(A) = \{\boldsymbol{x} \mid A\boldsymbol{x} = 0\} \subseteq \mathbb{R}^n$ is called the *nullspace* (or *kernel*) of A. In other words, $\ker(A)$ is simply the set of all solutions to the homogeneous system $A\boldsymbol{x} = \mathbf{0}$.
- The set

$$\ker(A^\top) = \{\boldsymbol{y} \mid A^\top\boldsymbol{y} = 0\} \subseteq \mathbb{R}^m$$

is called the *lefthand nullspace* of A because $\ker(A^\top)$ is the set of all solutions to the left-hand homogeneous system $\boldsymbol{y}^\top A = \mathbf{0}^\top$.

Spanning the Nullspace To determine a spanning set for $\ker(A)$, where $\mathrm{rank}(A_{m\times n}) = r$, row reduce A to a row echelon form U, and solve $U\boldsymbol{x} = \mathbf{0}$ for the basic variables in terms of the free variables to produce the general solution of $A\boldsymbol{x} = \mathbf{0}$ in the form

$$\boldsymbol{x} = x_{f_1}\boldsymbol{h}_1 + x_{f_2}\boldsymbol{h}_2 + ... + x_{f_{n-r}}\boldsymbol{h}_{n-r}.$$

By definition, the set $\mathcal{H} = \{\boldsymbol{h}_1, \boldsymbol{h}_2, ..., \boldsymbol{h}_{n-r}\}$ spans $\ker(A)$. Moreover, it can be proven that $\mathcal{H}$ is unique in the sense that $\mathcal{H}$ is independent of the row echelon form U.

7. Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$ and consider a linear system of equations $A\boldsymbol{x} = \boldsymbol{b}$. (a) Explain why $A\boldsymbol{x} = \boldsymbol{b}$ is consistent if and only if $b \in \mathrm{im}(A)$. (b) Explain why a consistent system $A\boldsymbol{x} = \boldsymbol{b}$ has a unique solution if and only if $\ker(A) = \{\boldsymbol{0}\}$.

8. Suppose that $A \in \mathrm{Mat}_{3\times 3}(\mathbb{R})$ such that

$$\mathcal{R} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \right\}, \quad \text{and} \quad \mathcal{N} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

span $\mathrm{im}(A)$ and $\ker(A)$, respectively, and consider a linear system $A\boldsymbol{x} = \boldsymbol{b}$, where $b = (1, -7, 0)^\top$. (a) Explain why $A\boldsymbol{x} = \boldsymbol{b}$ must be consistent. (b) Explain why $A\boldsymbol{x} = \boldsymbol{b}$ cannot have a unique solution.

9. If $A = \begin{bmatrix} -1 & 1 & 1 & -2 & 1 \\ -1 & 0 & 3 & -4 & 2 \\ -1 & 0 & 3 & -5 & 3 \\ -1 & 0 & 3 & -6 & 4 \\ -1 & 0 & 3 & -6 & 4 \end{bmatrix}$ and $A = \begin{bmatrix} -2 \\ -5 \\ -6 \\ -7 \\ -7 \end{bmatrix}$ is $b \in \mathrm{im}(A)$?

10. Suppose that $A \in \mathrm{Mat}_{n\times n}(\mathbb{R})$. (a) If $\mathrm{im}(A) = \mathbb{R}^n$, explain why A must be nonsingular. (b) If A is nonsingular, describe its four fundamental subspaces.

11. Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$. Show that $\ker(A) = \{\boldsymbol{0}\}$ if and only if $\mathrm{rank}(A) = n$.

Zero Nullspace If A is an $m \times n$ matrix, then
- $\ker(A) = \{\boldsymbol{0}\}$ if and only if $\mathrm{rank}(A) = n$.
- $\ker(A^\top) = \{\boldsymbol{0}\}$ if and only if $\mathrm{rank}(A) = m$.

Left-Hand Nullspace If $\mathrm{rank}(A_{m\times n}) = r$, and if $PA = U$, where P is nonsingular and U is in row echelon form, then the last $m - r$ rows in P span the left-hand nullspace of A. In other words, if $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$, where P_2 is $(m-r) \times m$, then

$$\ker(A^\top) = \mathrm{im}(P_2^\top).$$

Equal Nullspaces For two matrices A and B of the same shape:
- $\ker(A) = \ker(B)$ if and only if $A \overset{red}{\sim} B$.
- $\ker(A^\top) = \ker(B^\top)$ if and only if $A \overset{kol}{\sim} B$.

12. Determine a spanning set for $\ker(A)$, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$.

13. Consider the matrices $A = \begin{bmatrix} 1 & 1 & 5 \\ 2 & 0 & 6 \\ 1 & 2 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -4 & 4 \\ 4 & -8 & 6 \\ 0 & -4 & 5 \end{bmatrix}$. (a) Do A and B have the same row space? (b) Do A and B have the same column space? (c) Do A and B have the same nullspace? (d) Do A and B have the same left-hand nullspace?

14. Determine a spanning set for $\ker(A^\top)$, where $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 1 & 4 \end{bmatrix}$.

15. Suppose $\mathrm{rank}(A_{m\times n}) = r$, and let $P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$, be a nonsingular matrix such that $PA = U = \begin{pmatrix} C_{r\times n} \\ \boldsymbol{0} \end{pmatrix}$, where U is in row echelon form. Prove

$$\mathrm{im}(A) = \ker(P_2).$$

16. Let $(4,3,2,1)^\top \in \mathbb{R}^4$ be a given vector and let $A = \begin{bmatrix} a & -1 & 0 & 0 \\ a & b & -1 & 0 \\ a & 0 & b & -1 \\ a & 0 & 0 & b \end{bmatrix}$ denote a given matrix. Discus (and carefully explain) for which values of parameters a and b we have $(4,3,2,1)^\top \in \mathrm{im}(A)$.

17. Consider vector subspace of $\mathbb{R}^4$ spanned by vectors $x_1 = (-1,0,1,2)^\top$, $x_2 = (1,2,-3,5)^\top$ and $x_3 = (1,4,0,9)^\top$. Find system of homogeneous linear equations for which space of solution is exactly subspace of $\mathbb{R}^n$ spanned by above three given vectors.

18. Explain is the set, which contains columns of matrix A, linearly independent set, if we have that

$$A = \begin{bmatrix} 7 & 3 & 0 & \dots & 0 & 0 \\ 2 & 7 & 3 & \dots & 0 & 0 \\ 0 & 2 & 7 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 7 & 3 \\ 0 & 0 & 0 & \dots & 2 & 7 \end{bmatrix}_{n\times n}$$

(solve the problem without computing $\det(A)$). Is the matrix A a singular matrix? (Recall: square matrix with no inverse is called a singular matrix.)

19. Find for which value of unknown x will vector $(0,1,1,4)^\top \in \mathbb{R}^4$ belong to $\mathrm{im}(A)$ if

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1-x & 1 & 1 & 1 \\ 0 & 1-x & 1 & 1 \\ 0 & 0 & 1-x & 1 \end{bmatrix}.$$

InC: 2, 5, 7, 8, 10, 13, 14. HW: 16, 17, 18, 19.